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Asymptotic Description of a Viscous Fluid Layer

Enrique Cerda,[1] René Rojas,[2, 3] and Enrique Tirapegui[4]

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We prove that the exact non local equation derived by the present authors for the temporal linear evolution of the surface of a viscous incompressible fluid reduces asymptotically for high viscosity to a second order Mathieu type equation proposed recently by Cerda and Tirapegui. The equation describes a strongly damped pendulum and the conditions of validity of the asymptotic regime are given in terms of the relevant physical parameters.

KEY WORDS: Faraday instability; viscous fluid.

When a layer of fluid of height h on an horizontal plate is vibrated vertically stationary waves appear in the free surface above some amplitude threshold (Faraday's instability). Let $\vec{x}=(x, y)$ be horizontal coodinates, z the vertical axis and we choose the origin in the free surface when the system is at rest (the plate is at $z=-h$). We call $\xi(\vec{x}, t)$ the vertical displacement of the free surface and $\xi_{\vec{k}}(t)$ its horizontal Fourier transform (we consider the system as infinitely extended in the horizontal directions). For an inviscid fluid Benjamin and Ursell[(1)] derived in the linear approximation a Mathieu equation for $\xi_{\vec{k}}(t)$ of the form (dots represent derivatives with respect to time)

$$\ddot{\xi}_{\vec{k}}(t)+\omega_k^2 \xi_{\vec{k}}(t)=0 \tag{1}$$

where $\omega_k^2=k \tanh(kh)(g+\tau k^2/\rho)$ is the usual dispersion relation for surface waves (g is gravity, τ the surface tension and ρ the density). Equation (1)

[1] Departamento de Física, Universidad de Santiago.
[2] Centro de Física No Lineal y Sistemas Complejos de Santiago.
[3] Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas de la Universidad de Chile.
[4] Departamento de Física, Facultad de Ciencias Físicas y Matemáticas de la Universidad de Chile.

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corresponds to an undamped pendulum and weak viscosity has been introduced phenomenologically adding a dissipative term to the Benjamin–Ursell (BU) Eq. (1). In these last years the case of strong viscosity has interested many authors. Numerical studies of the Navier–Stokes equations have been done[2, 3] and compared with experiments. However no simple equation playing here the role of the corrected BU equation had been proposed. Recently Cerda and Tirapegui (CT) derived such an equation[4–6] for realistic boundary conditions. The CT equation is again a Mathieu type equation which corresponds to a strongly damped pendulum with coefficients which differ from the BU equation valid for weak viscosity. The Cerda–Tirapegui equation describes asymptotically for strong viscosity the linear evolution of the free surface of the fluid and our purpose here is to give a formal proof of the validity of this asymptotic description.

We shall first summarize the derivation of the CT equation for $\xi_{\vec{k}}(t)$ when the fluid is at rest and then we show how to add the effect of the vibration of the plate (all details can be found in our previous work[5, 6]). The starting point is the derivation of an exact equation for $\xi_{\vec{k}}(t)$ which is non local in time due to memory effects. We begin the Navier–Stokes equation for the velocity $\vec{v}(\vec{x}, z, t) = (v_1, v_2, v_3)$ with the correct boundary conditions for a viscous fluid. We insist in this point since a non local equation for $\xi_{\vec{k}}(t)$ has been derived recently[9] but imposing unphysical boundary conditions in the bottom of the plate which restricts the validity of the equation to the case of deep water. We impose in the upper surface the usual kinematical conditions relating $\vec{v}$ with $\xi(\vec{x}, t)$ and the equality of forces

$$\sum_{l=1}^{3} T_{jl} n_l = (p_o + \tau(1/R_1 + 1/R_2))\, n_j, \qquad j = 1, 2, 3$$

where p_o is the atmospheric pressure, (R_1, R_2) the radius of curvature in the two horizontal directions, $\hat{n} = (n_1, n_2, n_3)$ the unitary normal to the surface and T_{jl} the stress tensor there. It has the form $T_{jl} = p\,\delta_{jl} - \rho\nu(\partial_j v_l - \partial_l v_j)$ where ν is the viscosity and p the pressure. On the plate we impose the no-slip boundary conditions $\vec{v}(\vec{x}, z = -h, t) = 0$. We write $\vec{v} = -\vec{\nabla}\phi + \vec{u}$ and call $\vec{u}(\vec{x}, z, t) = (u_1, u_2, u_3)$ the diffusive velocity since it will contain the boundary effects depending on the viscosity which can not be realized by a potential term. After linearizing around the static solution ($\vec{v} = 0$, $p_{st} = p_o - \rho g z$) of the Navier–Stokes equation we prove that linearly one has

$$(\partial_t + 2\nu k^2)^2\, \xi_{\vec{k}}(t) + \omega_k^2 \xi_{\vec{k}}(t) + \frac{(\partial_t + 2\nu k^2)}{\cosh kh}\, u_{3\vec{k}}(z, t)\big|_{z=-h} + 2\nu k \tanh kh\, \partial_z u_{3\vec{k}}(z, t)\big|_{z=0} = 0 \qquad (2)$$

This exact equation relates $\xi_{\vec{k}}(t)$ to the horizontal Fourier transform $u_{3\vec{k}}(z, t)$ of the vertical component $u_3(\vec{x}, z, t)$ of the diffusive velocity. The function $u_{3\vec{k}}$ satisfies the equation

$$[\partial_t - \nu(\partial_z^2 - k^2)]\, u_{3\vec{k}}(z, t) = 0 \tag{3}$$

with the boundary conditions

$$u_{3\vec{k}}(z, t)\,|_{z=0} = -2\nu k^2 \xi_{\vec{k}}(t) \tag{4}$$

$$[\sinh kh\, u_{3\vec{k}}(z, t) + k \cosh kh\, u_{3\vec{k}}(z, t)]\,|_{z=0} = -k(\partial_t + 2\nu k^2)\, \xi_{\vec{k}}(t)$$

Equation (2) will become our exact equation for $\xi_{\vec{k}}(t)$ when $u_{3\vec{k}}(z, t)$ is expressed as a functional of $\xi_{\vec{k}}(t)$ solving Eqs. (3) and (4). The physical interpretation of (2) is clear: the effects of the,two boundaries correspond to the two last terms in (2) and the other additional term with respect to the non dissipative BU equation (1) is the translation $\partial_t \to \partial_t + 2\nu k^2$ of the time derivative which we can identify with the effect of dissipation by friction in the region of potential motion of the fluid.[5] Since we are interested in the long time behavior we solve Eq. (3) taking initial conditions at time $t_o \to -\infty$ and we obtain

$$u_{3\vec{k}}(t) = \int_{-\infty}^{\infty} dt' \exp[-\nu k^2(t-t')]\, K(t-t', z)\, \xi_{\vec{k}}(t') \tag{5}$$

where the kernel $K(t, z)$ can be calculated exactly[5, 6] and the non local character in time due to the memory effects of the autonomous equation for $\xi_{\vec{k}}(t)$ obtained replacing (5) in (2) is now explicit. We can write this exact non local equation formally as an infinite series in the time derivatives $\partial_t^{(n)} = \partial^n/\partial t^n$, $n \geqslant 1$. The result is

$$\sum_{n \geqslant 1} \tilde{c}_n\, \partial_t^{(n)} \xi_{\vec{k}}(t) + \omega_k^2 \xi_{\vec{k}}(t) = 0 \tag{6}$$

where the coefficients $\tilde{c}_n$ will be specified below. If the plate is vibrated vertically its coordinate z_p is $z_p(t) = -h + A\varphi(u = \Omega t)$, where $\varphi(u)$ has period 2π, $\max |\varphi(u)| = O(1)$. The effective gravity $g_e(t)$ in the reference system in which the plate is at rest is (points represent derivatives with respect to time)

$$g_e(t) = g + \ddot{z}_p(t) = g(1 + \Gamma\chi(\Omega t)) \tag{7}$$

where $\chi(u) = \varphi''(u)$ (primes are derivatives with respect to u) and $\Gamma = A\Omega^2/g$. The equation for this situation is obtained from (2) or (6) replacing g by $g_e(t)$ in the expression of ω_k^2. This changes ω_k^2 to $\omega_k(t)^2$ given by

$$\omega_k(t)^2 = \omega_k^2(1 + \Gamma_k \chi(\Omega t)), \qquad \Gamma_k = \frac{\Gamma}{1 + \tau k^2/\rho g} \tag{8}$$

The coefficients $\tilde{c}_n$ in (6) are $(v \equiv kh)$

$$\tilde{c}_n = \frac{(-1)^n}{(\nu k^2)^{n-2}} c_n(v), \qquad n \geqslant 1 \tag{9}$$

$$c_n(v) = \sum_{p \geqslant 1} \frac{a_p(v)}{u_p(v)^{n-1}}, \qquad n \geqslant 3 \tag{10}$$

$$c_2(v) = 1 + \sum_{p \geqslant 1} \frac{a_p(v)}{u_p(v)} \equiv \mathscr{F}(v) \leqslant \frac{3}{2} \tag{11}$$

$$c_1(v) \equiv -2\mathscr{G}(v) \tag{12}$$

Here $\mathscr{F}(v)$ is of order $O(1)$ for all v while $\mathscr{G}(v) \approx 1$ for $v \gtrsim 1$, $\mathscr{G}(v) \approx 3/2v^2$ for $v \ll 1$, $a_p(v)$ are bounded dimensionless positive functions of v and $(p \geqslant 1)$

$$u_p(v) = 1 + \frac{\alpha_p(v)}{v^2}, \qquad p^2\pi^2 < \alpha_p(v) < ((2p+1)/2)^2 \, \pi^2 \tag{13}$$

where $\alpha_p(v)$ are functions of v. The explicit expressions of all the preceding functions can be found in refs. 5 and 6. Putting $\xi_k = \exp(st)$ in (6) the values of s are determined by the dispersion relation (see also refs. 2 and 7) $DR(s) = \widetilde{F}(s) + \omega_k^2$ where

$$\widetilde{F}(s) = (s + 2\nu k^2)^2 + \nu^2 k \, \frac{q^2 + k^2}{\cosh v} \, \frac{4qk \sinh v - (q^2 + k^2) \sinh(qh)}{k \sinh(qh) \cosh v - q \sinh v \cosh(qh)}$$
$$+ 4\nu^2 k^3 \tanh v \, \frac{q \sinh(qh) \sinh v - k \cosh(qh) \cosh v}{k \sinh(qh) \cosh v - q \sinh v \cosh(qh)} \tag{14}$$

with $q \equiv \sqrt{s/\nu + k^2}$. In mathematical terms we have in (6) a pseudodifferential operator and its symbol is the funtion $DR(s)$[8] and we remark that properties (9)–(13) are direct consequences of a Mittag–Loefler expansion[5, 6] of $\widetilde{F}$. We consider now Eq. (6) with forcing (ω_k^2 is replaced by

$\omega_k(t)^2$ given by (8)): putting $u = \Omega t$, $\partial_t = \Omega \partial_u$, and multiplying by $(\nu k^2)^{-2}$ we can write (6) in the dimensionless form

$$\sum_{n \geqslant 1} (-\eta)^n \, c_n(v) \, \partial_u^{(n)} \xi_k(u) + c_o(u, v) \, \xi_k(u) = 0 \tag{15}$$

$$c_o(u, v) = \frac{\omega_k(t)^2}{(\nu k^2)^2} = \frac{\tanh v(1 + \beta v^2)}{\alpha v^3} (1 + \Gamma_k \chi(u)) \tag{16}$$

$$\varepsilon \equiv \frac{\Omega h^2}{\nu}, \qquad \alpha \equiv \frac{\nu^2}{g h^3}, \qquad \beta \equiv \frac{\tau}{\rho g h^2}, \qquad \eta \equiv \frac{\varepsilon}{v^2} \tag{17}$$

In the previous equations the dependence on the dimensionless parameters $(\varepsilon, \alpha, \beta, v, \Gamma)$ is explicitly exihibited. The Cerda–Tirapegui equation is obtained from (15) keeping only the first three terms, i.e., up to $N = 2$ in the sum. In what follows we shall derive the mathematical conditions which allow the use of the troncated equation near the instabilities. If we put $p = -\eta \partial_u$ Eq. (15) can be written

$$H(u, p = -\eta \partial_u) \, \xi_k(u) = 0$$

with

$$H(u, p) = \sum_{n \geqslant 1} c_n(v) \, p^n + c_o(u, v) \tag{18}$$

Our interest is in the asymptotic behavior for $\eta \ll 1$. We make the WKB type ansatz

$$\begin{gathered} \xi_k = \mathrm{e}^{-1/\eta \, \phi(u, \eta)} \\ \phi(u, \eta) = \phi^{(0)}(u) + \eta \phi^{(1)}(u) + \eta^2 \phi^{(2)}(u) + \cdots \end{gathered} \tag{19}$$

Replacing in (15) we obtain a hierarchy of equations for $\{\phi^{(j)}(u)\}$. The leading behavior is given by the functions $\phi^{(0)}(u)$ and $\phi^{(1)}(u)$ which satisfy the equations

$$H(u, \partial_u \phi^{(0)}(u)) = 0 \tag{20}$$

$$\left(\left. \frac{\partial H(u, p)}{\partial p} \right|_{p = \partial_u \phi^{(o)}} \right) \partial_u \phi^{(1)} = \frac{1}{2} \left(\left. \frac{\partial^2 H(u, p)}{\partial p^2} \right|_{p = \partial_u \phi^{(0)}} \right) \partial_u^{(2)} \phi^{(0)} \tag{21}$$

The coefficients $c_n(v)$ in $H(u, p)$ are given by (10)–(12) and since $u_1(v) < u_2(v) < \cdots < u_j(v) < \cdots$ one has ($\mu \equiv u_1(v)^{-1}$)

$$c_n(v) \leqslant \mu^{n-2} \sum_{p \geqslant 1} \frac{a_p(v)}{u_p(v)}, \qquad n \geqslant 3 \tag{22}$$

Due to (13) $u_1(v) > 1 + (\alpha_1^{\min}/v^2)$, and then $\mu < 1/(1 + (\alpha_1^{\min}/v^2)) < 1$ for $v \lesssim 1$. Notice that μ increases if v increases but one always has $\mu < 1$. We shall consider now μ as an small expansion parameter and we see then that we can expect the leading term to be a good quantitative approximation for $v \lesssim 1$ while for big v we can expect at least qualitative agreement with the complete non local equation (15). Using (22) we can write

$$c_n(v) = \mu^{n-2} b_n(v), \qquad n \geqslant 3 \tag{23}$$

with $0 < b_n(v) < 1/2$ due to (22) and (11). Expanding $\phi^{(j)}(u)$, $j \geqslant 1$, in the form

$$\phi^{(j)}(u) = \phi_0^{(j)}(u) + \mu \phi_1^{(j)}(u) + \mu^2 \phi_2^{(j)}(u) + \cdots \tag{24}$$

and replacing in formulas (20), (21) we obtain for the leading terms $(\phi_0^{(0)}(u), \phi_0^{(1)}(u))$ the equations

$$c_2(\partial_u \phi_0^{(0)})^2 + c_1(\partial_u \phi_0^{(0)}) + c_0(u, v) = 0 \tag{25}$$

$$(c_1 + 2c_2\, \partial_u \phi_0^{(0)})\, \partial_u \phi_0^{(1)} = \tfrac{1}{2} c_2\, \partial_u^{(2)} \phi_0^{(0)} \tag{26}$$

The solution of (15) in this double asymptotics (η and μ small) is then

$$\xi_{\vec{k}} = \exp\left[-\frac{1}{\eta} (\phi_0^{(0)}(u) + \eta \phi_0^{(1)}(u)) \right] \tag{27}$$

We recall now that the Cerda–Tirapegui equation is obtained troncating Eqs. (6) or (15) at $n = 2$. From (15) one gets the CT equation

$$\eta^2 c_2(v)\, \partial_u^{(2)} \xi_{\vec{k}} + \eta c_1(v)\, \partial_u \xi_{\vec{k}} + c_0(u, v)\, \xi_{\vec{k}} = 0 \tag{28}$$

We make for (28) a WKB ansatz writing

$$\xi_{\vec{k}} = \exp\left[-\frac{1}{\eta} (\psi^{(0)}(u) + \eta \psi^{(1)}(u) + \cdots) \right] \tag{29}$$

and replacing in (28) we find immediately that the functions $\psi^{(0)}(u)$ and $\psi^{(1)}(u)$ satisfy Eqs. (25) and (26) thus showing that the CT equation gives the correct asymptotic behavior of the complete non local equation (6) when η and μ are small quantities (we recall that μ is always smaller than 1 and approaches this value only in the limit $v \to \infty$). This ends then our discussion of the conditions of validity of CT equation.

We go now to a second part of this paper in which we shall discuss how the previous conditions of validity can be expressed in terms of physical parameters. The CT equation is

$$\ddot{\xi}_{\vec{k}}(t) + 2\bar{\gamma}_k \dot{\xi}_{\vec{k}}(t) + \bar{\omega}_k^2 (1 + \Gamma_k \chi(\Omega))\, \xi_{\vec{k}}(t) = 0 \tag{30}$$

$$\bar{\gamma}_k = \frac{\nu k^2 \mathscr{G}(kh)}{2\mathscr{F}(kh)}, \qquad \bar{\omega}_k = \frac{\omega_k^2}{\mathscr{F}(kh)}, \qquad \Gamma_k = \frac{\Gamma}{1 + \tau k^2/\rho g} = \frac{\Gamma}{1 + \beta v^2} \tag{31}$$

If we write (30) with the same parameters $(\varepsilon, \alpha, \beta, v, \Gamma)$ appearing in Eq. (15) we have

$$\partial_u^{(2)} \xi_{\vec{k}} + \frac{v^2 \mathscr{G}(v)}{\varepsilon \mathscr{F}}\, \partial_u \xi_{\vec{k}} + \frac{v(1 + \beta v^2) \tanh v}{\alpha \varepsilon^2 \mathscr{F}(v)} \left(1 + \frac{\Gamma}{1 + \beta v^2}\, \chi(u)\right) \xi_{\vec{k}} = 0 \tag{32}$$

Our strategy will be to study the CT equation (from now on we shall always speak of the CT equation) and to determine the parameters $(\varepsilon, \alpha, \beta)$ defined in (17) for which the instability arises and the threshold value Γ_c of the control parameter and the critical wave number k_c. Once k_c is known we know $v_c = k_c h$ and we can determine $\eta = \varepsilon/v_c^2$ and check the two conditions for the validity of (30), i.e., $\eta = \varepsilon/v_c^2 \ll 1$, $\mu = (1 + \alpha_1(v_c)/v_c^2)^{-1} \ll 1$, with $\pi^2 < \alpha_1(v_c) < \frac{9}{4}\pi^2$. If $\eta \lesssim 1$, and since μ is strictly smaller than one for all finite v_c, we can expect at least qualitative validity of the CT equation for the purpose of the determination of the instability threshold. We consider first the equation for a damped pendulum written in the form

$$\ddot{\xi}(t) + 2\gamma \dot{\xi}(t) + \omega^2 (1 + \Gamma^{(o)} \chi(\Omega t))\, \xi(t) = 0 \tag{33}$$

We can obtain exact results in the study of (33) if we take for $\chi(u)$ a step function of period 2π defined by $\chi(u) = 1$, $-\pi/2 \leqslant u < \pi/2$; $\chi(u) = -1$, $\pi/2 \leqslant u < 3\pi/2$, and we can certainly expect that the scenario obtained by this special choice of $\chi(u)$ will be the general one. We put $\hat{\gamma} \equiv \gamma/\omega$, $\sigma \equiv \omega/\Omega$ and notice that in the analysis of the behavior of Eq. (33) it is useful to distinguish two cases:

(a) $\Gamma^{(o)} > |1 - \hat{\gamma}^2|$ which is related to strong damping (strong viscosity in the Faraday problem) since $\Gamma^{(o)} > 1$ for $\hat{\gamma} = 0$;

(b) $\Gamma^{(o)} < |1-\hat{\gamma}^2|$ in which case one can show that when $\hat{\gamma}$ goes to zero the threshold value of $\Gamma^{(o)}$ also vanishes as it must for weak damping. We focus in $\Gamma^{(o)} > |1-\hat{\gamma}^2|$. The relevant quantity in the analysis is ($\tilde{\sigma} \equiv \sigma\sqrt{\Gamma^{(o)} - 1 + \hat{\gamma}^2}$, $\tilde{k} \equiv \sigma\sqrt{\Gamma^{(o)} + 1 - \hat{\gamma}^2}$)

$$X = \cosh(\pi\tilde{\sigma})\cos(\pi\tilde{k}) + \frac{\tilde{\sigma}^2 + \tilde{k}^2}{2\tilde{\sigma}\tilde{k}}\sinh(\pi\tilde{\sigma})\sin(\pi\tilde{k}) \tag{34}$$

where $(\tilde{\sigma}, \tilde{k})$ are real quantities since $\Gamma^{(o)} > |1-\hat{\gamma}^2|$. The Floquet analysis of (33) tells us that the condition for instability is $|X| \geqslant \cosh(2\pi\sigma\hat{\gamma})$. We consider the "resonant curves" $\tilde{\mathscr{C}}(n)$ defined by $\tilde{k} = n$ which will be represented in the $(\Gamma^{(o)}, \sigma)$ plane by the equations $(n = 1, 2, 3,\dots)$

$$\Gamma^{(o)} = \frac{n^2}{\sigma^2} - 1 + \hat{\gamma}^2 \equiv \tilde{f}_n(\sigma) \tag{35}$$

On the curve $\tilde{\mathscr{C}}(n)$ we have $X = (-1)^n\cosh(\pi\tilde{\sigma})$ and the condition for instability $|X| \geqslant \cosh(2\pi\sigma\hat{\gamma})$ tells us that in each resonant curve we must have

$$\Gamma^{(o)} \geqslant 1 + 3\hat{\gamma}^2 \equiv \tilde{f}(\sigma) \tag{36}$$

in order to be unstable and since $\tilde{f}(\sigma)$ does not depend on σ the marginal curve $\tilde{\mathscr{C}}$ given by $\Gamma^{(o)} = 1 + 3\hat{\gamma}^2$ is a line parallel to the σ axis in the $(\Gamma^{(o)}, \sigma)$ plane. Since X changes sign between the curves $\tilde{\mathscr{C}}_n$ and $\tilde{\mathscr{C}}_{n+1}$ there must be a curve in between for which $X = 0$. This tells us that we will have disjoint tongues of instability surrounding each resonant curve and it can be checked that the minimums of the tongues are almost in the line $\Gamma^{(o)} = 1 + 3\hat{\gamma}^2$ (for simplicity we shall say they are on this line). We use now these results in Eq. (30) and we see that $(\hat{\gamma}, \Gamma^{(o)}, \sigma)$ become $(\bar{\gamma}_k/\bar{\omega}_k, \Gamma_k, \bar{\omega}_k/\Omega)$ with

$$\frac{\bar{\gamma}_k^2}{\bar{\omega}_k^2} = \frac{\alpha v^3 \mathscr{G}(v)^2}{\mathscr{F}(v)\tanh(v)(1+\beta v^2)} \tag{37}$$

$$\frac{\Omega^2}{\bar{\omega}_k^2} = \alpha\varepsilon^2 \frac{\mathscr{F}(v)}{v\tanh(v)(1+\beta v^2)} \tag{38}$$

We consider now in the (Γ, v) plane the resonant curves $\mathscr{C}(n)$ (corresponding to $\Gamma^{(o)} = \tilde{f}_n(\sigma)$) and the marginal curve $\mathscr{C}$ (corresponding to $\Gamma^{(o)} = 1 + 3\hat{\gamma}^2$). One has

$$\Gamma = (1+\beta v^2)\left(n^2\varepsilon^2 \frac{\alpha}{1+\beta v^2}\frac{\mathscr{F}(v)}{v\tanh(v)} + \frac{\alpha}{1+\beta v^2}\frac{v^3\mathscr{G}(v)^2}{\mathscr{F}(v)\tanh(v)} - 1\right)$$
$$\equiv f_n(v;\alpha,\beta,\varepsilon) \tag{39}$$

$$\Gamma = 1+\beta v^2 + 2\alpha H(v) \equiv f(v;\alpha,\beta),\ H(v) \equiv \frac{3}{2}\frac{v^3\mathscr{G}(v)^2}{\mathscr{F}(v)\tanh(v)} \tag{40}$$

where (39) are the curves $\mathscr{C}(n)$ and (40) the curve $\mathscr{C}$. These curves are shown in Fig. 1 for the values ($\alpha = 0.133$, $\beta = 1.437$, $\varepsilon = 9.856$). The minimum (v^*, Γ^*) of the marginal curve is obtained from the equation $f'(v) = 0$ which gives

$$\frac{\beta}{\alpha} = -\frac{H'(v)}{v} \equiv \tilde{H}(v) \tag{41}$$

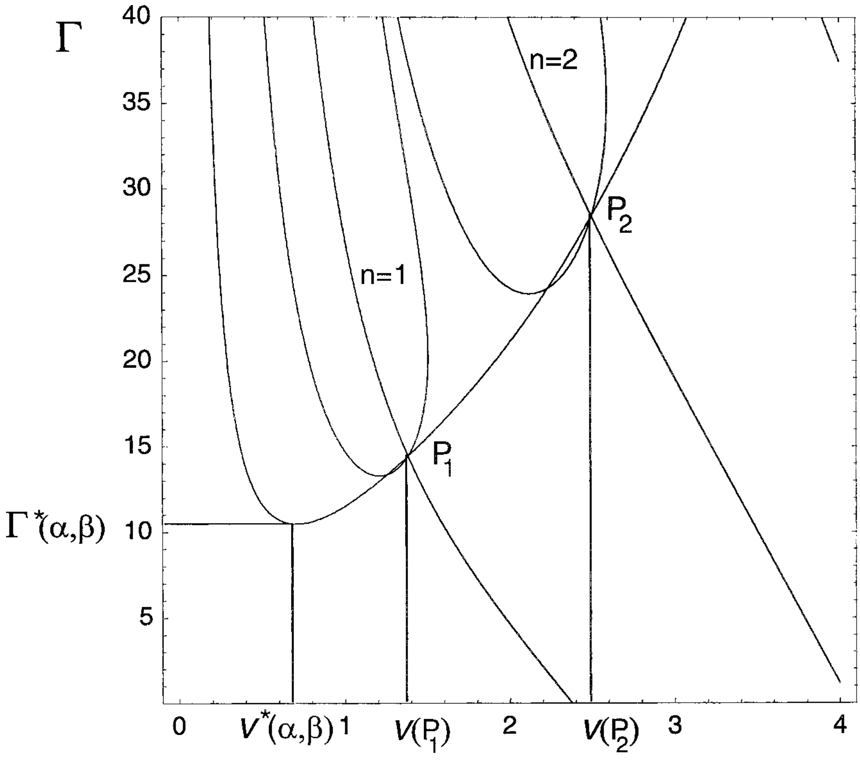

Fig. 1. The marginal curve with its minimum (v^*, Γ^*) and the two first resonant curves.

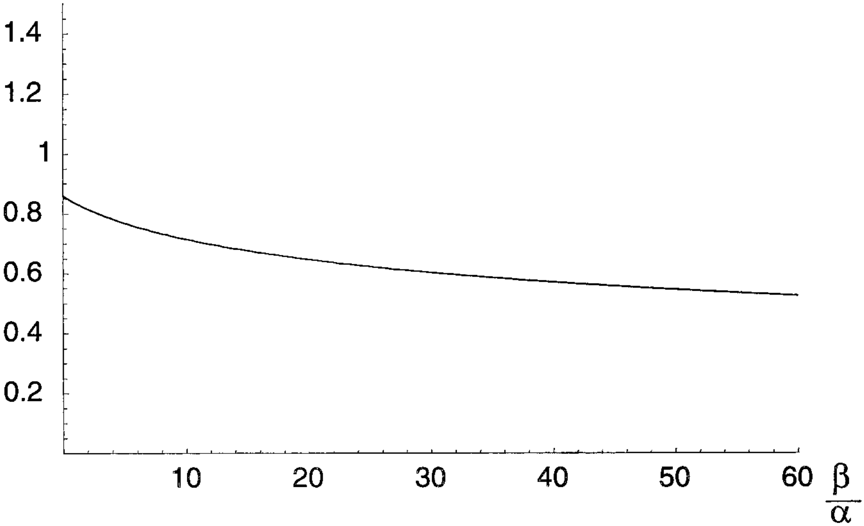

Fig. 2. The curve $v^*(\beta/\alpha)$.

Solving this equation in which $\tilde{H}(v)$ is a function only of v, we obtain the minimum $v^*(\beta/\alpha)$ in Fig. 1. We have drawn the function $v^*(\beta/\alpha)$ in Fig. 2. This function decreases monotously and tends to zero as $(\beta/\alpha)^{-1/4}$ for big (β/α).

We can determine the values $(v(P_1), v(P_2),...)$ of Fig. 1 eliminating Γ between (39) and (40). This gives

$$n^2\varepsilon^2 = 2\left[\frac{v^4\mathscr{G}(v)^2}{\mathscr{F}(v)^2} + \frac{1+\beta v^2}{\alpha}\,\frac{v\tanh(v)}{\mathscr{F}(v)}\right] \equiv M(v;\alpha,\beta) \tag{42}$$

We have drawn this function $M(v;\alpha,\beta)$ for $(\alpha = 0.133,\ \beta = 1.437)$ in Fig. 3. The value m_0 of $M(v;\alpha,\beta)$ for $v = 0$ and the leading behavior for $v \to \infty$ are both independent of (α, β) and one has $m_0 = 25/8$ and $M(v;\alpha,\beta) \approx \frac{8}{9}v^4$ when $v \to \infty$. Then when (α, β) are given we calculate $v^*(\beta/\alpha)$ from Fig. 2 and using Fig. 3 we determine the important value $m(\alpha, \beta)$. We have to distinguish two cases:

(a) If $\varepsilon^2 < m(\alpha,\beta)$ we have that $v_c \approx v^*(\beta/\alpha) \leqslant 0.86$ and since $v^*(\beta/\alpha)$ decreases slowly (as $(\beta/\alpha)^{-1/4}$) we have that $v_c = O(1)$ unless (β/α) is very big, and then $\eta = \varepsilon/v_c^2 \approx \varepsilon$ and the CT equation will be a quantitative approximation if $\varepsilon \ll 1$ and we can expect qualitative agreement when ε becomes $O(1)$. In this case the second parameter $\mu < (1 + \alpha_1^{\min}/v_c^2) \approx (1+\pi^2)^{-1} \approx 10^{-1}$ is always small;

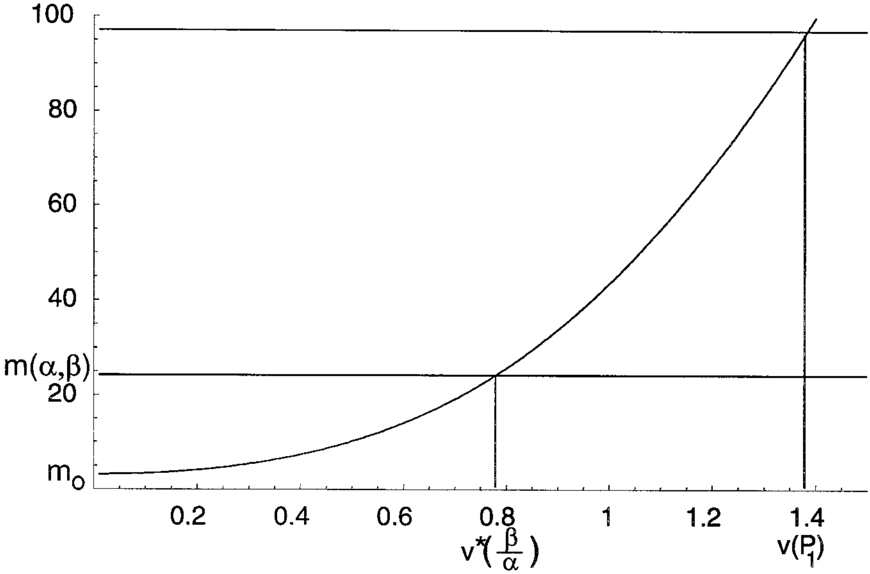

Fig. 3. The function $M(v; \alpha, \beta)$ for $\alpha = 0.133$, $\beta = 1.437$.

(b) If $\varepsilon^2 > m(\alpha, \beta)$ one will have $v_c = v(P_1) > v^*(\beta/\alpha)$ (this is the situation drawn in Fig. 3) and one can ask the question of the behavior of $\eta = \varepsilon/v_c^2$ for big ε. The answer is given by Eq. (42) which tells us that $(\varepsilon/v_c^2)^2 \approx 8/9$ when $\varepsilon \to \infty$. We see then that for big ε the parameter $\eta \lesssim 1$ and one will also have $\mu < 1$ (although it can be very near one if v_c is big enough) and consequently we may expect that the CT equation will be valid at least qualitatively.

We can summarize our analysis plotting η as a function of ε for different values of (α, β) as it is done in Fig. 4 where we can see that the CT equation can be a qualitative description for almost all values of ε since η is of $O(1)$, except for a small interval around the point Q', or much smaller than one when ε is very small. The values of (α, β) for which the points Q and Q' approach each other are the most favorable to have $\eta \lesssim O(1)$ in almost all the range of variation of ε. We gave an explicit relation between α and β for this to occur in the frame of the WKB solution of the CT equation in ref. 5. In conclusion we can have $\eta \lesssim 1$ and $\mu < 1$ for any ε but the strict condition for validity of the CT equation, i.e., $\eta \ll 1$, is only verified for ε small enough. The parameter $\eta = \Omega/\nu k_c^2 = \varepsilon/v_c^2$ which controls the validity of the CT equation corresponds to $(l/\delta)^2$ where $\delta = (\nu/\Omega)^{1/2}$ and l is the penetration length of the motion defined in refs. 5 and 6 since when $\varepsilon^2 < m(\alpha, \beta)$ one has $\eta \approx \varepsilon = \Omega h^2/\nu$ and $l = h$ and when $\varepsilon^2 > m(\alpha, \beta)$ one has

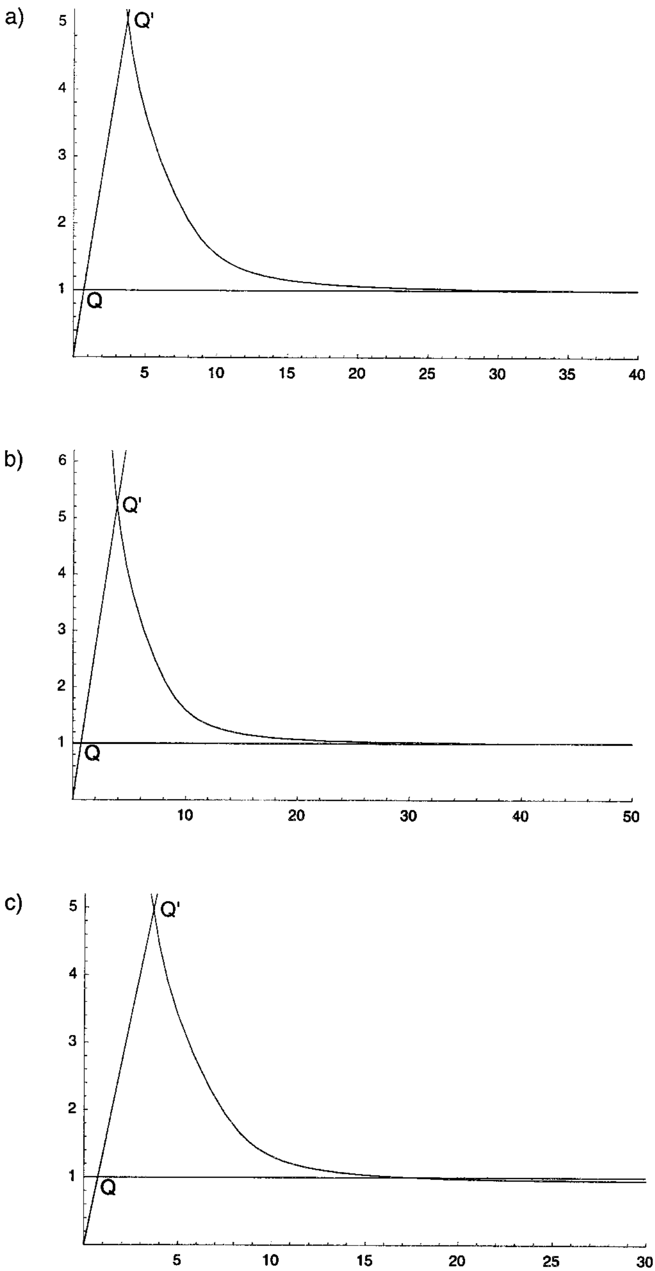

Fig. 4. The parameter η as a function of ε for (a) ($\alpha = 2.5$, $\beta = 1$), (b) ($\alpha = 10$, $\beta = 5$), (c) ($\alpha = 1000$, $\beta = 5$).

$v_c > v^*(\beta/\alpha) = 0.86$ and as soon as $v_c > 1$ we can estimate $l \approx k_c^{-1}$ and then $\eta = (l/\delta)^2$.

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